A note on converse Lyapunov-Krasovskii theorems for nonlinear neutral systems in Sobolev spaces

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Abstract: The problem of existence of a Lyapunov-Krasovskii functional (LKF) for nonlinear neutral type time-delay systems is revisited considering the uniform stability analysis and the LKF in a Sobolev space of absolutely continuous functions with bounded derivatives.

Keywords: Neutral time-delay systems, Lyapunov-Krasovskii functional, Stabilty.

1. INTRODUCTION

Analysis of stability of dynamical systems is a key point in different domains of science and technology, and especially in the control systems area Khalil (2015). The main method for analysis of stability is based on application of Lyapunov functions or Lyapunov-Krasovskii functionals Khalil (2015); Hale (1977); Kolmanovsky and Nosov (1986); Kolmanovskii and Myshkis (1999). Despite that for ordinary differential equations a significant progress has been obtained, and the first necessary and sufficient results for asymptotic stability have been proposed in 1950th and for (integral) input-to-state stability in 1990th, for time-delay systems these results started to appear only recently Pepe and Karafyllis (2013); Lin and Wang (2018a,b). In particular, considering the nonlinear neutral type time-delay systems in the Hale's form, the papers Pepe and Karafyllis (2013); Pepe et al. (2017) present the equivalent conditions of asymptotic and exponential stability in terms of existence of LKF. The goal of this note is to develop this result for a more generic class of neutral time-delay systems (skipping the restriction on the Hale's form). Differently from Pepe and Karafyllis (2013); Pepe et al. (2017), where the stability notions have been analyzed in the space $\mathbb{W}^{1,+\infty}_{[-\tau,0]}$ (the Sobolev space of continuous functions with essentially bounded derivatives), while also frequently the space $\mathbb{W}^{1,2}_{[-\tau,0]}$ is used Fridman (2014) (the derivative is square integrable), in this work we will carry out the mathematical treatment in $\mathbb{W}^{1,1}_{[-\tau,0]}$ (the Sobolev space of continuous functions with integrable derivatives) and the LKF will be also defined in this space. The technical advantages of such a change consists in the established Lipschitz continuity of the solutions of the systems with locally Lipschitz continuous right-hand side in $\mathbb{W}^{1,1}_{[-\tau,0]}$ (in $\mathbb{W}^{1,+\infty}_{[-\tau,0]}$ such a property has been proven for the systems in the Hale's form only). We will also consider the case of uniform stability and convergence in the presence of an exogenous input taking the values in a compact set.

The outline of this note is as follows. Some preliminary results are introduced in section 2. The problem statement is given in Section 3. The main result is established in Section 4.

2. PRELIMINARIES

Denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of real and natural numbers, respectively, and $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$.

Some conventional results used in this paper are as follows:

Lemma 1. Khalil (2015) Jensen's inequality. Let $f : [a, b] \to \mathbb{R}_+$ be an integrable function for $[a, b] \subset \mathbb{R}$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a *concave* function. Then

$$\varphi\left(\frac{1}{b-a}\int_a^b f(s)ds\right) \geq \frac{1}{b-a}\int_a^b \varphi\left(f(s)\right)ds.$$

Lemma 2. Khalil (2015) Gronwall's Lemma. For $T > 0$ assume that $\phi : [0, T] \to \mathbb{R}_+$ is a bounded measurable function, $C : [0, T] \to \mathbb{R}_+$ is an integrable function, $B \geq 0$, and

$$\phi(t) \leq B + \int_0^t C(s)\phi(s)ds \quad \forall t \in [0, T].$$

Then

$$\phi(t) \leq B\exp\left(\int_0^t C(s)ds\right) \quad \forall t \in [0, T].$$

2.1 Definitions of norms and spaces

For a Lebesgue measurable function of time $d : [a, b] \to \mathbb{R}^m$, $[a, b] \subset \mathbb{R}$ define the norm $\|d\|_{[a,b]} = \text{ess sup}_{t\in[a,b]}|d(t)|$, where $|\cdot|$ is the standard Euclidean norm in $\mathbb{R}^m$, then $||d||_\infty = \text{ess sup}_{t\geq 0}|d(t)|$ and the space of d with $\|d\|_{[a,b]} < +\infty$ ($||d||_\infty < +\infty$) we further denote as $\mathcal{L}^m_{[a,b]}$ ($\mathcal{L}^m_\infty$).

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Denote by $\mathbb{C}^n_{[a,b]}$, $[a,b]\subset\mathbb{R}$ the Banach space of continuous functions $\phi:[a,b]\to\mathbb{R}^n$ with the uniform norm $\|\phi\|_{[a,b]}=\sup_{a\le s\le b}|\phi(s)|$; and by $\mathbb{W}^{1,p}_{[a,b]}$, $p\in\mathbb{N}$ and $\mathbb{W}^{1,\infty}_{[a,b]}$ the Sobolev spaces of absolutely continuous functions $\phi:[a,b]\to\mathbb{R}^n$ with bounded derivatives having the respective norms $\|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}}=\|\phi\|_{[a,b]}+\left(\int_a^b|\dot\phi(s)|^pds\right)^{\frac{1}{p}}<+\infty$ and $\|\phi\|_{\mathbb{W}^{1,\infty}_{[a,b]}}=\|\phi\|_{[a,b]}+\|\dot\phi\|_{[a,b]}<+\infty$, where $\dot\phi(s)=\frac{\partial\phi(s)}{\partial s}$ (it is a Lebesgue measurable essentially bounded function for $\phi\in\mathbb{W}^{1,\infty}_{[a,b]}$, *i.e.* $\dot\phi\in\mathcal{L}^n_{[a,b]}$) [1].

Lemma 3. For any $\phi\in\mathbb{W}^{1,\infty}_{[a,b]}$ and $p\in\mathbb{N}\cup\{+\infty\}$ the following inequalities are satisfied:

$$\min\{1,(b-a)^{\frac{1}{p}-1}\}\|\phi\|_{\mathbb{W}^{1,1}_{[a,b]}}\le\|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}}\le\max\{1,(b-a)^{\frac{1}{p}}\}\|\phi\|_{\mathbb{W}^{1,\infty}_{[a,b]}}.$$

Proof. By the norm definition we deduce:

$$\begin{aligned}\|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}}&=\|\phi\|_{[a,b]}+\left(\int_a^b|\dot\phi(s)|^pds\right)^{\frac{1}{p}}\\&\le\|\phi\|_{[a,b]}+\left(\int_a^b\|\dot\phi\|^p_{[a,b]}ds\right)^{\frac{1}{p}}\\&\le\max\{1,(b-a)^{\frac{1}{p}}\}\|\phi\|_{\mathbb{W}^{1,\infty}_{[a,b]}},\end{aligned}$$

and using Lemma 1:

$$\begin{aligned}\|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}}&=\|\phi\|_{[a,b]}+\left(\int_a^b|\dot\phi(s)|^pds\right)^{\frac{1}{p}}\\&\ge\|\phi\|_{[a,b]}+(b-a)^{\frac{1}{p}-1}\int_a^b|\dot\phi(s)|ds\\&\ge\min\{1,(b-a)^{\frac{1}{p}-1}\}\|\phi\|_{\mathbb{W}^{1,1}_{[a,b]}}\end{aligned}$$

that was necessary to prove.

2.2 Comparison functions and their properties

A continuous function $\sigma:\mathbb{R}_+\to\mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0)=0$; it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded.

Lemma 4. For any $\alpha\in\mathcal{K}$ there exists a continuous function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ admitting the following properties: $\gamma(0)=0$, $\gamma(s)>0$ for all $s>0$, and

$$\gamma(s)\le\alpha(s),\ |\gamma(s)-\gamma(s')|\le|s-s'|\quad\forall s,s'\in\mathbb{R}_+.$$

In addition,

$$\lim_{s\to+\infty}\gamma(s)=+\infty$$

provided that $\alpha\in\mathcal{K}_\infty$.

Proof. As in Pepe and Karafyllis (2013), consider

$$\gamma(s)=\min_{0\le y\le s}\{\alpha(y)+s-y\},$$

then $\gamma(0)=0$, and for any $s>0$ the sum $\alpha(y)+s-y$ is always positive for any $y\in[0,s]$, therefore $\gamma(s)>0$ for all $s>0$. If $\alpha\in\mathcal{K}_\infty$, then $\lim_{s\to+\infty}\gamma(s)=\lim_{s\to+\infty}\min_{0\le y\le s}\{\alpha(y)+s-y\}=+\infty$. The property $\gamma(s)\le\alpha(s)$ for all $s\in\mathbb{R}_+$ follows from the definition of γ for $y=s$. Finally, take any $s,s'\in\mathbb{R}_+$ such that they possess a relation $s\le s'$, then

$$\begin{aligned}|\gamma(s)-\gamma(s')|=&|\min_{0\le y\le s}\{\alpha(y)+s-y\}\\&-\min_{0\le y\le s'}\{\alpha(y)+s'-y\}|\\\le&\max_{0\le y\le s'}|(\alpha(y)+s-y)-(\alpha(y)+s'-y)|\\=&|s-s'|,\end{aligned}$$

that was necessary to prove.

A continuous function $\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot,r)\in\mathcal{K}$ and $\beta(r,\cdot)$ is a strictly decreasing to zero for any fixed $r\in\mathbb{R}_+$.

Lemma 5. Sontag (1998) For any $\beta\in\mathcal{KL}$ there exist $\theta_1,\theta_2\in\mathcal{K}_\infty$ such that

$$\beta(s,t)\le\theta_1\left(\theta_2(s)e^{-t}\right)\quad\forall s\ge0,\ t\ge0.$$

2.3 Neutral time-delay systems

Consider a nonlinear functional differential equation of neutral type Kolmanovsky and Nosov (1986):

$$\dot x(t)=f(x_t,\dot x_t,d(t))\quad t\ge0 \tag{1}$$

where $x(t)\in\mathbb{R}^n$ and $x_t\in\mathbb{C}^n_{[-\tau,0]}$ is the state function, $x_t(s)=x(t+s)$, $-\tau\le s\le0$, with $\dot x_t\in\mathcal{L}^n_{[-\tau,0]}$; $d\in\mathcal{D}=\{d\in\mathcal{L}^m_\infty:||d||_\infty\le1\}$ is the external input; $f:\mathbb{C}^n_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]}\times\mathbb{R}^m\to\mathbb{R}^n$ is a continuous function, that is Lipschitz in the second variable with a constant smaller than 1, ensuring forward uniqueness and existence of the system solutions at least locally in time Kolmanovsky and Nosov (1986). We assume $f(0,0,d)=0$ for any $d\in D=\{d\in\mathbb{R}^m:|d|\le1\}$. For the initial function $x_0\in\mathbb{C}^n_{[-\tau,0]}$ and an input $d\in\mathcal{D}$ denote a unique solution of the system (1) by $x(t,x_0,d)$, which is an absolutely continuous function defined on some time interval $[-\tau,T)$ for $T>0$, then $x_t(x_0,d)\in\mathbb{C}^n_{[-\tau,0]}$ represents the corresponding state function, and $x_t(s,x_0,d)=x(t+s,x_0,d)$ for all $-\tau\le s\le 0$.

Given a continuous functional $V:\mathbb{R}_+\times\mathbb{C}^n_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]}\to\mathbb{R}_+$ define:

$$\begin{aligned}D^+V(t,\phi,\dot\phi,d)=\limsup_{h\to0^+}\frac{1}{h}[&V(t+h,x_h(\phi,d),\dot x_h(\phi,d))\\&-V(t,\phi,\dot\phi)],\end{aligned}$$

where $x_h(\phi,d)$ is a solution of the system (1) for $\phi\in\mathbb{W}^{1,\infty}_{[-\tau,0]}$ and $d\in D$ is a constant.

2.4 Stability definitions

In the following we summarize the well-known results from Kolmanovskii and Myshkis (1999); Fridman (2014) on stability of (1).

Definition 6. Pepe and Jiang (2006); Fridman et al. (2008); Lin and Wang (2018b) The system (1) is called uniformly globally asymptotically stable (uGAS), if for all $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$ and $d\in\mathcal{D}$ there exists $\beta\in\mathcal{KL}$ such that

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\le\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t)\quad\forall t\ge0.$$

[1] In Kolmanovsky and Nosov (1986); Fridman et al. (2008) the norm with $p=2$ has been only used for the state space of time-delay systems.

Instead of $\mathbb{W}^{1,1}_{[-\tau,0]}$ any other space $\mathbb{W}^{1,p}_{[-\tau,0]}$ can be used in this definition for an integer $p \in \mathbb{N} \cup \{+\infty\}$.

Obviously, in this case

$$|x(t, x_0, d)| \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t) \quad \forall t \ge 0,$$

as it is usually assumed Pepe and Jiang (2006); Fridman et al. (2008). Recall that in El'sgol'ts and Norkin (1973), under the assumption that f is Lipschitz in the second variable $\dot\phi$ with a constant smaller than 1, it is established that the latter estimate is equivalent to the stability in $\mathbb{W}^{1,\infty}_{[-\tau,0]}$.

Definition 7. A continuous functional $V : \mathbb{R}_+ \times \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}_+$ is called simple if $D^+V(t, \phi, \dot\phi, d)$ is independent on $\ddot\phi$.

For instance, a locally Lipschitz functional $V : \mathbb{C}^n_{[-\tau,0]} \to \mathbb{R}_+$ is simple.

Definition 8. Pepe and Jiang (2006); Fridman et al. (2008); Lin and Wang (2018b) A continuous functional $V : \mathbb{R}_+ \times \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}_+$ is called LKF for the system (1) if it is simple, there exist $p \in \mathbb{N} \cup \{+\infty\}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{K}$ such that V is Lipschitz continuous on bounded sets in $\mathbb{W}^{1,p}_{[-\tau,0]} \setminus \{0\}$, and for all $t \in \mathbb{R}_+$, $d \in D$ and $\phi \in \mathbb{W}^{1,p}_{[-\tau,0]}$:

$$\alpha_1(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}) \le V(t, \phi, \dot\phi) \le \alpha_2(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}),$$
$$D^+V(t, \phi, \dot\phi, d) \le -\alpha(V(t, \phi, \dot\phi)).$$

Note that existence of such a LKF implies that for all $t \in \mathbb{R}_+$, $d \in D$ and $\phi \in \mathbb{W}^{1,p}_{[-\tau,0]}$:

$$\alpha_1(|\phi(0)|) \le V(t, \phi, \dot\phi) \le \alpha_2(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}),$$
$$D^+V(t, \phi, \dot\phi, d) \le -\hat\alpha(|\phi(0)|),$$

where $\hat\alpha(s) = \alpha(\alpha_1(s))$ is a function from class $\mathcal{K}$, which is the standard LKF formulation used to establish asymptotic stability Fridman (2014).

Theorem 9. Fridman et al. (2008); Fridman (2014) If there exists a LKF for the system (1), then it is uGAS.

There exist also some converse results to Theorem 9, see *e.g.* Pepe and Karafyllis (2013), which are obtained for $V : \mathbb{C}^n_{[-\tau,0]} \to \mathbb{R}_+$ and a special class of f in the Hale's form.

3. PROBLEM STATEMENT

The goal of this work is to propose a converse of Theorem 9 for $x_t \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and for a more general class of f than in Pepe and Karafyllis (2013); Pepe et al. (2017). In particular the following hypothesis are accepted in the sequel.

Assumption 1. The system (1) is uGAS in the sense of Definition 9 with $p = 1$ and $f : \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \times \mathbb{R}^m \to \mathbb{R}^n$ is uniformly Lipschitz continuous on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]}$: for any closed and bounded subset $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$ there exists $L_\Upsilon > 0$ such that

$$|f(\phi, \dot\phi, d) - f(\varphi, \dot\varphi, d)| \le L_\Upsilon \|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $\phi, \varphi \in \Upsilon$ and $d \in D$.

An example of such a system is given by dynamics with distributed delays:

$$f(\phi, \dot\phi, d) = F(\phi, \int_{-\tau}^{0} \dot\phi(s)ds, d)$$

with a locally Lipschitz continuous function $F : \mathbb{C}^n_{[-\tau,0]} \times \mathbb{R}^{2n} \to \mathbb{R}^n$. From the relations between the norms given in Lemma 3 we observe that boundedness of x_t in $\mathbb{W}^{1,+\infty}_{[-\tau,0]}$ implies immediately a similar property for all other norms. Therefore, a stability analysis performed in the space $\mathbb{W}^{1,1}_{[-\tau,0]}$ seems to be less restrictive, which is the motivation for selection of that space in this problem formulation. However, as we will see later, in our case the stability and convergence in $\mathbb{W}^{1,1}_{[-\tau,0]}$ also imply the same properties in $\mathbb{W}^{1,p}_{[-\tau,0]}$ for all $p \in \mathbb{N} \cup \{+\infty\}$.

4. MAIN RESULTS

In this section, first, some preliminary results are established, which clarify the features and significance of the imposed assumptions, and second, a converse result is presented.

4.1 Auxiliary properties

Under the introduced restrictions we have the following useful property for the system (1):

Proposition 10. Let Assumption 1 be satisfied. Then in (1) for any $T > 0$ and a closed and bounded subset $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$ there exists $M_{T,\Upsilon} > 0$ such that

$$\|x_t(\phi, d) - x_t(\varphi, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le M_{T,\Upsilon}\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \quad \forall t \in [0, T],$$

for all $\phi, \varphi \in \Upsilon$ and $d \in \mathcal{D}$.

Proof. According to Assumption 1 there is $\beta \in \mathcal{KL}$ such that

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t) \quad \forall t \ge 0$$

for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in D$. For any Υ as above there exists $\rho > 0$ such that $\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \rho$ and $\|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \rho$, hence, we have

$$\max\{\|x_t(\phi, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, \|x_t(\varphi, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\} \le \beta(\rho, 0)$$

and

$$|f(x_t(\phi), \dot x_t(\phi), d(t)) - f(x_t(\varphi), \dot x_t(\varphi), d(t))|$$
$$\le L_{\beta(\rho,0)}\|x_t(\phi, d) - x_t(\varphi, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $t \ge 0$, where $L_{\beta(\rho,0)} > 0$ represents the Lipschitz constant of f on the set $\{\phi \in \mathbb{W}^{1,1}_{[-\tau,0]} : \|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\rho, 0)\}$. For any $\phi \in \mathbb{W}^{1,1}_{[-\tau,0]}$ we have:

$$x(t, \phi, d) = x(0, \phi, d) + \int_0^t f(x_s(\phi), \dot x_s(\phi), d(s))ds$$

for all $t \ge 0$, and for $\phi, \varphi \in \Upsilon$:

$$|x(t, \phi, d) - x(t, \varphi, d)| \le |x(0, \phi, d) - x(0, \varphi, d)|$$
$$+ \int_0^t |f(x_s(\phi), \dot x_s(\phi), d(s)) - f(x_s(\varphi), \dot x_s(\varphi), d(s))|ds$$
$$\le |x(0, \phi, d) - x(0, \varphi, d)|$$
$$+ L_{\beta(\rho,0)} \int_0^t \|x_s(\phi, d) - x_s(\varphi, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} ds.$$

Therefore,

$$\|x_t(\phi,d)-x_t(\varphi,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} = \sup_{-\tau\le s\le 0}|x_t(s,\phi,d)-x_t(s,\varphi,d)|$$
$$+\int_{-\tau}^{0}|\dot{x}_t(s,\phi,d)-\dot{x}_t(s,\varphi,d)|ds$$
$$\le |x(0,\phi,d)-x(0,\varphi,d)|$$
$$+\sup_{-\tau\le s\le 0}\int_0^{t+s}|\dot{x}(\sigma,\phi,d)-\dot{x}(\sigma,\varphi,d)|d\sigma$$
$$+\int_{-\tau}^{0}|\dot{x}_t(s,\phi,d)-\dot{x}_t(s,\varphi,d)|ds$$
$$\le |x(0,\phi,d)-x(0,\varphi,d)|$$
$$+\int_0^t|f(x_\sigma(\phi),\dot{x}_\sigma(\phi),d(\sigma))-f(x_\sigma(\varphi),\dot{x}_\sigma(\varphi),d(\sigma))|d\sigma$$
$$+\int_{t-\tau}^t|f(x_\sigma(\phi),\dot{x}_\sigma(\phi),d(\sigma))-f(x_\sigma(\varphi),\dot{x}_\sigma(\varphi),d(\sigma))|ds$$
$$\le |x(0,\phi,d)-x(0,\varphi,d)|$$
$$+2L_{\beta(\rho,0)}\int_{-\tau}^t\|x_\sigma(\phi,d)-x_\sigma(\varphi,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}d\sigma.$$

Next, using Lemma 2 we obtain:

$$\|x_t(\phi,d)-x_t(\varphi,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\exp(2L_{\beta(\rho,0)}(t+\tau))$$

for all $t\ge 0$. Consequently, take any $T>0$ then

$$\|x_t(\phi,d)-x_t(\varphi,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le M_{T,\Upsilon}\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \quad \forall t\in[0,T]$$

for $M_{T,\Upsilon}=\exp(2L_{\beta(\rho,0)}(T+\tau))$.

Thus, under Assumption 1 for the solutions of the system (1), for any bounded set of initial conditions and a compact interval of time, there is a kind of local Lipschitz property with respect to the initial conditions uniformly in d. It is worth to highlight that the result is substantiated in the space $\mathbb{W}^{1,1}_{[-\tau,0]}$, and a similar conclusion in $\mathbb{W}^{1,\infty}_{[-\tau,0]}$ was obtained for the system (1) in the Hale's form (Pepe and Karafyllis (2013); Pepe et al. (2017)).

Corollary 11. Let Assumption 1 be satisfied. Then for any $p\in\mathbb{N}\cup\{+\infty\}$, $x_t(x_0,d)\in\mathbb{W}^{1,p}_{[-\tau,0]}$ and for all $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$, $d\in\mathcal{D}$ and all $t\ge 0$, there exists $\beta_p\in\mathcal{KL}$ such that

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,p}_{[-\tau,0]}} \le \beta_p(\|x_0\|_{\mathbb{W}^{1,p}_{[-\tau,0]}},t) \quad \forall t\ge 0.$$

Proof. Under Assumption 1 there is $\beta\in\mathcal{KL}$ such that

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t) \quad \forall t\ge 0$$

for all $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$ and all $d\in\mathcal{D}$, which by the norm definition implies that

$$\|x_t(x_0,d)\|_{[-\tau,0]} \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t) \quad \forall t\ge 0$$

and

$$|\dot{x}(t,x_0,d)| = |f(x_t(x_0),\dot{x}_t(x_0),d(t))|$$
$$\le L_{\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},0)}\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$
$$\le L_{\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},0)}\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t) \quad \forall t\ge 0.$$

It is possible to show that $L_{\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},0)}$ can be selected as a non-decreasing function of $\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$ (*e.g.*, see Lemma A.4 in Pepe and Karafyllis (2013)). According to Lemma 5, there exist $\theta_1,\theta_2\in\mathcal{K}_\infty$ such that

$$\beta(s,t)\le\theta_1\left(\theta_2(s)e^{-t}\right) \quad \forall s\ge 0,\ t\ge 0.$$

Since $|\dot{x}(t,x_0,d)| \le L_{\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},0)}\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},0) < +\infty$ for all $t\ge 0$, then $x_t(x_0,d)\in\mathbb{W}^{1,+\infty}_{[-\tau,0]}$ for all $t\ge 0$, and

$$\|\dot{x}_t(x_0,d)\|_{[-\tau,0]} = \sup_{s\in[-\tau,0)}|\dot{x}(t+s,x_0,d)|$$
$$\le L_{\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},0)}\sup_{s\in[-\tau,0)}\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t+s)$$
$$\le L_{\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},0)}\theta_1(\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})e^{-t}e^{\tau})$$
$$\le \tilde{\beta}(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t) \quad \forall t\ge 0,$$

where $\tilde{\beta}(s,t)=L_{\beta(s,0)}\theta_1(\theta_2(s)e^{-t}e^{\tau})$ is a new function from the class $\mathcal{KL}$. Recall the result of Lemma 3 for any integer $p\in\mathbb{N}\cup\{+\infty\}$:

$$\min\{1,\tau^{\frac{1}{p}-1}\}\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \|x_0\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}$$
$$\le \max\{1,\tau^{\frac{1}{p}}\}\|x_0\|_{\mathbb{W}^{1,\infty}_{[-\tau,0]}},$$

then

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,\infty}_{[-\tau,0]}} \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t)+\tilde{\beta}(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t)$$
$$\le \beta_\infty(\|x_0\|_{\mathbb{W}^{1,\infty}_{[-\tau,0]}},t) \quad \forall t\ge 0,$$

where $\beta_\infty(s,t)=\beta(\max\{1,\tau\}s,t)+\tilde{\beta}(\max\{1,\tau\}s,t)\in\mathcal{KL}$ as needed. Finally, for any integer $p\in\mathbb{N}\cup\{+\infty\}$ using Jensen's inequality from Lemma 1 we obtain

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,p}_{[-\tau,0]}} = \|x_t(x_0,d)\|_{[-\tau,0]}$$
$$+\left(\int_{-\tau}^0|\dot{x}(t+s,x_0,d)|^p ds\right)^{\frac{1}{p}}$$
$$\le \|x_t(x_0,d)\|_{[-\tau,0]}+\left(\int_{-\tau}^0\|\dot{x}_t(x_0,d)\|^p_{[-\tau,0]}ds\right)^{\frac{1}{p}}$$
$$= \|x_t(x_0,d)\|_{[-\tau,0]}+\tau^{1/p}\|\dot{x}_t(x_0,d)\|_{[-\tau,0]}$$
$$\le \max\{1,\tau^{1/p}\}\|x_t(x_0,d)\|_{\mathbb{W}^{1,\infty}_{[-\tau,0]}}$$
$$\le \max\{1,\tau^{1/p}\}[\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t)+\tilde{\beta}(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t)]$$
$$\le \beta_p(\|x_0\|_{\mathbb{W}^{1,p}_{[-\tau,0]}},t) \quad \forall t\ge 0,$$

where $\beta_p(s,t) = \max\{1,\tau^{1/p}\}(\beta(\max\{1,\tau^{1-\frac{1}{p}}\}s,t)+\tilde{\beta}(\max\{1,\tau^{1-\frac{1}{p}}\}s,t))$ is from class $\mathcal{KL}$. This also implies that $x_t(x_0,d)\in\mathbb{W}^{1,p}_{[-\tau,0]}$ for all $t\ge 0$ and any $p\in\mathbb{N}\cup\{+\infty\}$.

Thus, we have demonstrated that under introduced hypothesis, the stability can be equivalently concluded in any space $\mathbb{W}^{1,p}_{[-\tau,0]}$ for an integer $p\in\mathbb{N}\cup\{+\infty\}$, and similarly for any closed and bounded subset $\Upsilon\subset\mathbb{W}^{1,1}_{[-\tau,0]}$:

$$|f(\phi,\dot{\phi},d)-f(\varphi,\dot{\varphi},d)| \le L_{\Upsilon,p}\|\phi-\varphi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}$$

for all $\phi,\varphi\in\mathbb{W}^{1,p}_{[-\tau,0]}$ and $d\in D$ such that $\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}} \le \min\{1,\tau^{\frac{1}{p}-1}\}\rho$ and $\|\varphi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}} \le \min\{1,\tau^{\frac{1}{p}-1}\}\rho$, where $L_{\Upsilon,p}=\max\{1,\tau^{1-\frac{1}{p}}\}L_\Upsilon$ with L_Υ given in Assumption

1 and $\rho > 0$ is the corresponding norm bound on Υ, *i.e.* the Lipschitz continuity of f in $\mathbb{W}^{1,p}_{[-\tau,0]}$ also follows.

4.2 Converse design of LKF

For brevity of exposition the analysis is presented in the space $\mathbb{W}^{1,1}_{[-\tau,0]}$ (we need the result of Proposition 10 formulated in this space).

Theorem 12. Let Assumption 1 be satisfied, then there exists a LKF for the system (1).

Proof. Under the introduced hypothesis and Lemma 5 there are $\theta_1, \theta_2 \in \mathcal{K}_\infty$ such that

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \theta_1\left(\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})e^{-t}\right) \quad \forall t \ge 0$$

for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{D}$, then by recalling Lemma 4, there exists a continuous, positive definite and radially unbounded function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ admitting the following properties:

$$\gamma(s) \le \theta_1^{-1}(s),\ |\gamma(s) - \gamma(s')| \le |s - s'| \quad \forall s, s' \in \mathbb{R}_+.$$

Now, for any $x_0 \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ select

$$V(x_0, \dot{x}_0) = \sup_{t\ge 0, d\in\mathcal{D}} \{\gamma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + t}{\kappa_2 + t}\}$$

with $\frac{\kappa_2}{1+\kappa_2} < \kappa_1 < \kappa_2 < +\infty$. Then

$$\begin{aligned} V(x_0, \dot{x}_0) &\le \sup_{t\ge 0, d\in\mathcal{D}} \{\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})e^{-t}\frac{\kappa_1 + t}{\kappa_2 + t}\} \\ &\le \frac{\kappa_1}{\kappa_2}\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \end{aligned}$$

and

$$\begin{aligned} V(x_0, \dot{x}_0) &\ge \sup_{t\ge 0, d\in\mathcal{D}} \{\gamma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1}{\kappa_2}\} \\ &\ge \gamma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1}{\kappa_2} \end{aligned}$$

since under introduced restrictions on κ_1 and κ_2 the functions $\frac{\kappa_1+t}{\kappa_2+t}$ and $e^{-t}\frac{\kappa_1+t}{\kappa_2+t}$ are strictly increasing and decreasing, respectively. Note that from this analysis $\gamma(s) \le \theta_2(s)$ for all $s \in \mathbb{R}_+$. Define $\underline{\gamma}(s) = \frac{s}{1+s}\inf_{\sigma\ge s}\gamma(\sigma)$, which is a function from class $\mathcal{K}_\infty$, then

$$\frac{\kappa_1}{\kappa_2}\underline{\gamma}(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \le V(x_0, \dot{x}_0) \le \frac{\kappa_1}{\kappa_2}\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$, and the LKF $V(x_0, \dot{x}_0)$ is coercive admitting a lower and an upper bounds in terms of functions from the class $\mathcal{K}_\infty$ with respect to $\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$ as desired. From the decreasing of $e^{-t}\frac{\kappa_1+t}{\kappa_2+t}$ it also follows that there exists $T^{x_0} > 0$ such that

$$V(x_0, \dot{x}_0) = \sup_{0\le t\le T^{x_0}, d\in\mathcal{D}} \{\gamma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + t}{\kappa_2 + t}\},$$

and since

$$\begin{aligned} \gamma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1}{\kappa_2} &\le V(x_0, \dot{x}_0) \\ &\le \sup_{0\le t\le T^{x_0}, d\in\mathcal{D}} \{\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})e^{-t}\}, \end{aligned}$$

then by the definition of T^{x_0} it has an upper estimate:

$$T^{x_0} \le \ln\left[\frac{\kappa_2}{\kappa_1}\frac{\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})}{\gamma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})}\right].$$

For any $0 < r < R < +\infty$ define a set

$$\Omega_{r,R} = \{x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]} : r \le \|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le R\},$$

consequently, there exists a finite $T^{r,R} = \sup_{x_0\in\Omega_{r,R}} T^{x_0}$, *i.e.* $T^{r,R} = \ln\left[\frac{\kappa_2}{\kappa_1}\frac{\theta_2(R)}{\gamma(r)}\right]$.

Let us check the Lipschitz continuity of V on any bounded and closed subset in $\mathbb{W}^{1,1}_{[-\tau,0]} \setminus \{0\}$ (here by $\{0\}$ we understand $x \in \mathbb{W}^{1,1}_{[-\tau,0]}$ with $\|x\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} = 0$). For any $x_0, x_1 \in \mathbb{W}^{1,1}_{[-\tau,0]} \setminus \{0\}$ denote $T^{x_0,x_1} = \max\{T^{x_0}, T^{x_1}\}$ and consider:

$$\begin{aligned} |V(x_1,\dot{x}_1) - V(x_0,\dot{x}_0)| &= |\sup_{0\le t\le T^{x_1}, d\in\mathcal{D}} \{\gamma(\|x_t(x_1,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+t}{\kappa_2+t}\} \\ &\quad - \sup_{0\le t\le T^{x_0}, d\in\mathcal{D}} \{\gamma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+t}{\kappa_2+t}\}| \\ &\le \sup_{0\le t\le T^{x_0,x_1}, d\in\mathcal{D}} \left|[\gamma(\|x_t(x_1,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) - \gamma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})]\frac{\kappa_1+t}{\kappa_2+t}\right| \\ &\le \sup_{0\le t\le T^{x_0,x_1}, d\in\mathcal{D}} \left|\gamma(\|x_t(x_1,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) - \gamma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\right| \\ &\le \sup_{0\le t\le T^{x_0,x_1}, d\in\mathcal{D}} \left|\|x_t(x_1,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} - \|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\right| \\ &\le \sup_{0\le t\le T^{x_0,x_1}, d\in\mathcal{D}} \|x_t(x_1,d) - x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, \end{aligned}$$

where on the last step and the step before the Lipschitz properties of the norm $\|\cdot\|_{\mathbb{W}^{1,\infty}_{[-\tau,0]}}$ and the function γ have been utilized, respectively. For any $x_0, x_1 \in \mathbb{W}^{1,1}_{[-\tau,0]} \setminus \{0\}$ there exist $0 < r < R < +\infty$ such that $x_0, x_1 \in \Omega_{r,R}$, then $T^{x_0,x_1} \le T^{r,R}$ and using Proposition 10 there exists $M_{T^{r,R},\Omega_{r,R}} > 0$ such that

$$\|x_t(x_1,d) - x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le M_{T^{r,R},\Omega_{r,R}}\|x_1 - x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $t \in [0, T^{r,R}]$ and all $d \in \mathcal{D}$, hence,

$$|V(x_1,\dot{x}_1) - V(x_0,\dot{x}_0)| \le M_{T^{r,R},\Omega_{r,R}}\|x_1 - x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $x_0, x_1 \in \Omega_{r,R}$ and any $0 < r < R < +\infty$,which implies the required Lipschitz continuity of V on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]} \setminus \{0\}$. The continuity at the origin follows from the upper estimate:

$$V(x_0,\dot{x}_0) \le \frac{\kappa_1}{\kappa_2}\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

that is satisfied for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$.

Finally, let us check the decreasing of the LKF V on the trajectories of the system (1) for $t > 0$ (recall that $\frac{\kappa_1+t}{\kappa_2+t}$ is a strictly increasing function of time):

$$\begin{aligned} V(x_t(x_0,d), \dot{x}_t(x_0,d)) &= \sup_{\sigma\ge 0, \delta\in\mathcal{D}} \{\gamma(\|x_\sigma(x_t(x_0,d),\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma}{\kappa_2+\sigma}\} \\ &= \sup_{\sigma\ge 0, \delta\in\mathcal{D}} \{\gamma(\|x_{t+\sigma}(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma}{\kappa_2+\sigma}\} \\ &= \sup_{\sigma\ge t, \delta\in\mathcal{D}} \{\gamma(\|x_\sigma(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma-t}{\kappa_2+\sigma-t}\} \\ &< \sup_{\sigma\ge t, \delta\in\mathcal{D}} \{\gamma(\|x_\sigma(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma}{\kappa_2+\sigma}\} \\ &\le \sup_{\sigma\ge 0, \delta\in\mathcal{D}} \{\gamma(\|x_\sigma(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma}{\kappa_2+\sigma}\} = V(x_0,\dot{x}_0), \end{aligned}$$

thus, V is strictly decreasing along the trajectories of (1).

The proposed expression for a LKF,

$$V(x_0,\dot{x}_0) = \sup_{t\ge 0, d\in\mathcal{D}} \{\gamma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+t}{\kappa_2+t}\},$$

has an explicit form and it is more simple than in Pepe and Karafyllis (2013); Pepe et al. (2017), where a two step procedure for construction has been proposed, and also a more restrictive class of f has been considered.

5. CONCLUSIONS

The problem of existence of a LKF for nonlinear neutral type time-delay systems is solved considering the uGAS stability property in $\mathbb{W}^{1,1}_{[-\tau,0]}$ space. It is shown that a uniform stability in this space also implies stability in $\mathbb{W}^{1,p}_{[-\tau,0]}$ for any $p \in \mathbb{N} \cup \{+\infty\}$, and that a Lipschitz property of f defined in such a space can be transformed in the same property of the solutions of (1). Finally, a simple expression of LKF defined in $\mathbb{W}^{1,1}_{[-\tau,0]}$ is given. Extension of these results to input-to-state stability property and non-autonomous systems are directions of future research.

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